Symmetric Extension of Overcomplete Tensor Decomposition via Koszul–Young Flattenings

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Abstract

This is an extension of the rank detection theorem proved in [KMW24]. Instead of considering tensors composed of rank-1 terms of the form $a \otimes b \otimes c$ for generic vectors a, b, c, we consider those of the form $a \otimes a \otimes a$.

1 Introduction

In [KMW24], Kothari, Moitra, and Wein introduced an efficient rank detecting algorithm for order-3 generic tensors T via the Koszul–Young flattening. In this writing piece, we extend this to symmetric order-3 tensors T. That is, we consider tensors T of the form

$$T = \sum_{\ell=1}^{r} a^{(\ell)} \otimes a^{(\ell)} \otimes a^{(\ell)},$$

where $a^{(1)}, \dots, a^{(r)} \in \mathbb{R}^n$ are generic vectors. The goal is to determine r, given the entries $T = (T_{ijk})$.

2 Rank Detection

The original rank detection method proposed in [KMW24] does not work in the symmetric case. We propose a similar but altered method here.

Suppose T is an $n \times n \times n$ tensor, and fix integers $p, q = 2p + 1$ with $1 \le p < q < n$. Since q depends on p, we will not mention q as an input variable. Define a matrix $M = M(T; p)$ with rows indexed by

$$\{(S, j) \,:\, S \subseteq [q],\, |S| = p,\, j \in [n - q]\},$$

columns indexed by

$$\{(U, k) \,:\, U \subseteq [q],\, |U| = p + 1,\, k \in [n - q]\},$$

and entries

$$M_{Sj,Uk} := \sum_{i=1}^{q} \mathbb{1}_{U = S \sqcup \{i\}} \cdot \sigma(U, i) \cdot T_{i,(j+q),(k+q)}. \tag{1}$$

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Here, $\mathbb{1}_A$ denotes the $\{0,1\}$-valued indicator for an event A, and $\sigma(U,i) \in \{\pm 1\}$ is the parity of i's position in U, that is,

$$\sigma(U,i) := (-1)^{|\{j \in U\,:\, j<i\}|}. \tag{2}$$

Note though it is written as a summation, at most one term in the sum in (1) is nonzero, which is when $U = S \sqcup \{i\}$.

Lemma 2.1. *Suppose T is a rank-1 tensor, i.e. $T = a \otimes a \otimes a$. Let $a = (a_1, \dots, a_n)$ and split a into $\widehat{a} = (a_1, \dots, a_q)$ and $\tilde{a} = (a_{q+1}, \dots a_n)$. Then the flattening $M = M(T;p)$ from (1) is the Kronecker product $M = A(\widehat{a};p) \otimes (\tilde{a}\tilde{a}^\top)$ where $A = A(\widehat{a};p)$ is the $\binom{q}{p} \times \binom{q}{p+1}$ matrix*

$$A_{SU} = \sum_{i=1}^{q} \mathbb{1}_{U=S\sqcup\{i\}} \cdot \sigma(U,i) \cdot a_i. \tag{3}$$

As a consequence, as long as $\tilde{a}$ is not the zero vector, M has the same rank as A.

The lemma below shows that the matrix A is rank-deficient, with generic rank $\binom{q-1}{p}$:

Lemma 2.2. *[KMW24, Lemma 3.1] Consider the matrix $A = A(\hat{a};p)$ defined in (3). Let $q = 2p+1$. If $a_1, \dots, a_q$ are all zero then $A = 0$ and so $rank(A) = 0$. Otherwise, $rank(A) = \binom{q-1}{p}$. Furthermore, if $a_i \neq 0$ for some $i \in [q]$ then the columns of A indexed by $\{U : i \in U\}$ form a basis for the column span of A, and similarly the rows $\{S : i \notin S\}$ form a basis for the row span.*

For proof of Lemma 2.2, see Section 3.1 of [KMW24].

3 Main results

Theorem 3.1 (Symmetric Tensor Rank Detection)**.** *Let $1 \leq p < q = 2p+1 < n$. If $T = \sum_{\ell=1}^{r} a^{(\ell)} \otimes a^{(\ell)} \otimes a^{(\ell)}$ is $n \times n \times n$ with generically chosen $a^{(\ell)}$'s, and*

$$r \leq \left(2 - \frac{1}{p+1}\right) n - (6p+2)$$

then the matrix $M(T;p)$ defined in (1) has rank exactly $r\binom{q-1}{p}$.

For intuition, assume $1 \ll p \ll n$. Then we have $(2 - \frac{1}{p+1}) \to 2$ as $p \to \infty$. Moreover, since we assume n grows much faster than p, then $6p+2$ is negligible compared to the leading n term. Therefore, the condition above is roughly $r \leq 2n$, asymptotically.

Proof. This proof will mostly follow proof of Theorem 2.4 in [KMW24], with a few adjustments to fit our symmetric case. We start by separating $a^{(\ell)} = (a_1^{(\ell)}, a_2^{(\ell)}, \dots, a_q^{(\ell)}, a_{q+1}^{(\ell)}, \dots, a_n^{(\ell)})$ into $\widehat{a}^{(\ell)} = (a_1^{(\ell)}, a_2^{(\ell)}, ..., a_q^{(\ell)})$ and $\tilde{a}^{(\ell)} = (a_{q+1}^{(\ell)}, \dots, a_n^{(\ell)})$. We are going to use $\widehat{a}^{(\ell)}$ to construct A. By Lemma 2.1, the equation for M becomes $M = \sum_{\ell=1}^{r} A(\widehat{a}^{(\ell)};p) \otimes (\tilde{a}^{(\ell)}\tilde{a}^{(\ell)\top})$.

Since a is generic, $a_i \neq 0$ for all $i \in [q]$. Then, the matrix $A = A(\widehat{a};p)$ defined in (3) can be factored as $A = \mathrm{diag}(v) \cdot \tilde{A} \cdot \mathrm{diag}(w)$ where $\tilde{A}_{SU} := A(\mathbb{1};p) = \sum_{i\in[q]} \mathbb{1}_{U=S\sqcup\{i\}} \cdot \sigma(U,i)$, $v_S = \prod_{i \in S} a_i^{-1}$, and $w_U = \prod_{i\in U} a_i$. By Lemma 2.2 we have $rank(\tilde{A}) = \binom{q-1}{p}$. Therefore, we can factor $\tilde{A} = \tilde{Q}\tilde{R}^\top$

where $\tilde{Q}, \tilde{R}$ are $\binom{q}{p}\times\binom{q-1}{p}$ and $\binom{q}{p+1}\times\binom{q-1}{p}$, respectively. Now write $A = QR^\top$ where $Q = \mathrm{diag}(v)\cdot\tilde{Q}$ and $R = \mathrm{diag}(w)\cdot\tilde{R}$. Factor M as

$$M = \sum_{\ell=1}^{r} A(\widehat{a}^{(\ell)}; p) \otimes (\tilde{a}^{(\ell)}\tilde{a}^{(\ell)\top}) = \begin{bmatrix} Q^{(1)} & & Q^{(r)} \\ \otimes & \cdots & \otimes \\ \tilde{a}^{(1)} & & \tilde{a}^{(r)} \end{bmatrix} \begin{bmatrix} R^{(1)} & & R^{(r)} \\ \otimes & \cdots & \otimes \\ \tilde{a}^{(1)} & & \tilde{a}^{(r)} \end{bmatrix}^\top =: Q'R'^\top \tag{4}$$

Note in the symmetric case, M is square, so each Kronecker product has dimension $\binom{q}{p}(n-q)\times\binom{q-1}{p}$. The equation above gives a factorization of M as the product of two matrices with inner dimension $r\binom{q-1}{p}$ and outer dimension $\binom{q}{p}(n-q) = \binom{q}{p+1}(n-q)$ (since $q = 2p+1$). To show M has rank exactly $r\binom{q-1}{p}$, we are going to argue it is enough to show that both Q' and R' have full column rank. Notice by the decomposition definition of rank of a matrix, $\mathrm{rank}(M) \leq r\binom{q-1}{p} =: R$. Moreover, if we assume Q' has full column rank and $R \leq \binom{q}{p}(n-q)$, the outer dimension of Q', then Q' has an $R\times R$ nonzero minor. A similar argument applies to R'. Thus, M has an $R\times R$ nonzero minor, which shows $\mathrm{rank}(M) \geq R$.

Let m be the smallest integer such that $r \leq mq$. Note we can increase number of columns of Q' to $r' = mq$ by adding newly generated generic copies of Kronecker product $Q^{(r+i)}\otimes\tilde{a}^{(r+i)}$. If we can show the enlarged matrix has linearly independent columns, then Q' also has linearly independent columns. Consider the first $r'\binom{q-1}{p}$ rows of the enlarged matrix and call it P; this is a square submatrix that uses all the columns, and it consists of an $m\times m$ grid of square blocks that each have dimension $(p+1)\binom{q}{p} = q\binom{q-1}{p}$. Each block contains scaled copies of Q in a $(p+1)\times q$ grid, and recall that Q has dimensions $\binom{q}{p}\times\binom{q-1}{p}$. Above, we needed the enlarged Q' to have more rows than columns. That is, we need $\binom{q}{p}(n-q) \geq r'\binom{q-1}{p} \Leftrightarrow \binom{q}{p}(n-q) \geq mq\binom{q-1}{p} \Leftrightarrow \binom{q}{p}(n-q) \geq m(p+1)\binom{q}{p} \Leftrightarrow n-q \geq m(p+1)$.

It now suffices to show $\det(P)$ is nonzero as a polynomial in the symbolic variables a. To do this, it suffices to show this statement is true after plugging in zero for certain variables $\tilde{a}_i^{(\ell)}$ such that only the m square blocks on the diagonal of P remain nonzero. That is, we only need to show the determinant is not zero for all square blocks that lie on the diagonal of P. Take one block

$$P' = \begin{bmatrix} \tilde{a}_{(1+q)1}Q^{(1)} & \tilde{a}_{(1+q)2}Q^{(2)} & \cdots & \tilde{a}_{(1+q),q}Q^{(q)} \\ \tilde{a}_{(2+q)1}Q^{(1)} & \tilde{a}_{(2+q)2}Q^{(2)} & & \vdots \\ \vdots & & \ddots & \\ \tilde{a}_{2q-p,1}Q^{(1)} & \cdots & & \tilde{a}_{2q-p,q}Q^{(q)} \end{bmatrix}.$$

Then, we argue the rest of the proof that shows $\det(P') \neq 0$ is identical to the argument in [KMW24]. Since a is generically chosen, then $\widehat{a}$ and $\tilde{a}$ are also both generic. Note $\tilde{a}$ functions as b, and $\widehat{a}$ functions as a in the original proof. Thus, the symmetry in the original tensor will not affect how we construct P'. Hence, the same argument of finding one unique nonzero monomial from $\det(P')$ to show $\det(P') \neq 0$ also works here in our case.

Similarly, we can use the same argument to conclude that R' has full column rank, provided $n-q \geq m(p+1)$.

Above, we needed two inequalities to hold: $r \leq mq$ and $n-q \geq m(p+1)$. That is,

$$\frac{r}{q} \leq m \leq \frac{n-q}{p+1}.$$

Note m needs to be an integer by construction, so it suffices to have

$$\frac{r}{q} \leq \frac{n-q}{p+1} - 1.$$

Multiplying by q on both sides and recalling $q = 2p+1$, we have

$$r \leq \frac{q(n-q)}{p+1} - q = (2 - \frac{1}{p+1})n - \frac{q(p+1+q)}{p+1},$$

which leads to

$$r \leq (2 - \frac{1}{p+1})n - (6p + \frac{p+2}{p+1}) \leq (2 - \frac{1}{p+1})n - (6p+2) \quad \forall\, p \geq 1.$$

$\square$

References

[KMW24] Pravesh K. Kothari, Ankur Moitra, and Alexander S. Wein. Overcomplete tensor decomposition via koszul–young flattenings. *Proceedings of the 56th Annual ACM Symposium on Theory of Computing (STOC)*, 2024.